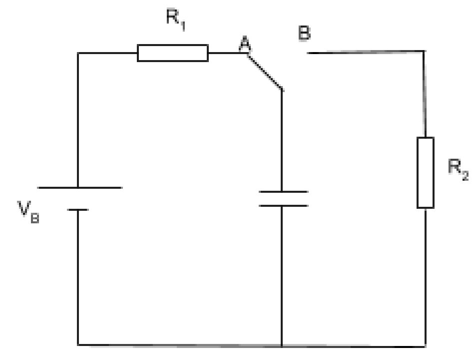
Derivation of Capacitor Charging and Discharging Equations

Charging of a Capacitor (Switch in Position A)



By Kirchhoff's Laws, $V_B - V_R - V_C = 0$. With some simple substitution, $V_B - iR - \frac{q}{C} = 0$. We know that the current describes the rate at which charge is increasing on the plate of the capacitor. This can be written as $i = \frac{dq}{dt}$.
This leads to $V_B - \frac{dq}{dt}R - \frac{q}{C} = 0$
With some algebra, this equation becomes
$\frac{V_B C - q}{RC} = \frac{dq}{dt}$
Separating the variables of *q* and *t* gives $\frac{dt}{RC} = \frac{dq}{V_B C - q}$
This form can be analysed by assuming $u = V_B C - q$ which leads to $du = -dq$
So $\frac{dt}{RC} = -\frac{du}{u}$.

Integrating both sides $\int \frac{dt}{RC} = \int -\frac{du}{u}$ leads to $\frac{t}{RC} + k = -ln(u)$ where *k* is some constant.

Substituting for *u* and multiplying by -1 leads to $-\frac{t}{RC} - k = ln(V_B C - q)$
So $e^{-\frac{t}{RC}} e^{-k} = V_B C - q$ but we know that at *t=0, q=0 so* $e^{-k} = V_B C$
Then $V_B C e^{-\frac{t}{RC}} = V_B C - q$
Solving for *q* gives

$$q = V_B C(1 - e^{-\frac{t}{RC}})$$

which is the equation that describes the charge on a capacitor during charging.
Since the potential difference across the capacitor is $V = \frac{q}{C}$,

$$V_C = V_B(1 - e^{-\frac{t}{RC}})$$

As stated earlier, $i = \frac{dq}{dt}$ so $i = \frac{d}{dt}(V_B C(1 - e^{-\frac{t}{RC}})) = (-\frac{1}{RC})V_B C(-e^{-\frac{t}{RC}})$. This simplifies to:

$$i = \frac{V_B}{R} e^{-\frac{t}{RC}}$$

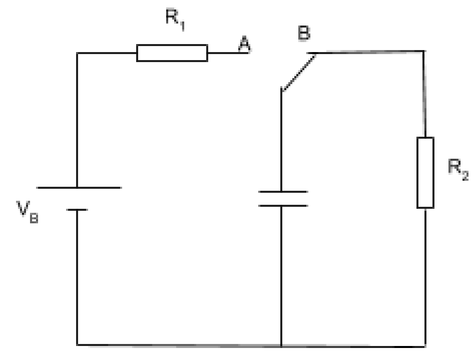
The resistor obeys Ohm's Law $V = iR$, so

$$V_R = V_B e^{-\frac{t}{RC}}$$

Notice that all of these equations include $e^{-\frac{t}{RC}}$. RC determines the rate of change for the curves. This is a quantity called the "time constant."

$$\tau = RC$$

<u>Discharging a Capacitor</u> <u>(Switch in Position B)</u>



When the switch is changed from position A to position B, Kirchhoff's law gives $V_C - V_R = 0$.
Subtituting $V_C = \frac{q}{C}$ and $V_R = iR$ leads to $\frac{q}{C} - iR = 0$.
During the discharge process, $i = -\frac{dq}{dt}$ so $\frac{q}{C} - (-\frac{dq}{dt})R = 0$.
Separating variables $-\frac{dt}{RC} = \frac{dq}{q}$.

Integrating both sides $\int -\frac{dt}{RC} = \int \frac{dq}{q}$ leads to $-\frac{t}{RC} + k = ln(q)$ and $e^{-\frac{t}{RC}} e^k = q$

Knowing that at *t=0*, $q=q_0$ produces the equation

$$q = q_0 e^{-\frac{t}{RC}}$$

Since $i = -\frac{dq}{dt}$ and $q_0 = CV_0$

$$i = \frac{V_0}{R} e^{-\frac{t}{RC}}$$

Going back to Ohm's law and Kirchhoff's law

$$V_C = V_R = V_0 e^{-\frac{t}{RC}}$$